BTZ-like black holes in even dimensional Lovelock theories

Fabrizio Canfora[1], Alex Giacomini[2].

[1]*Centro de Estudios Cientificos (CECS), Casilla 1469 Valdivia, Chile.*

[2]*Instituto de Fisica, Facultad de Ciencias, Universidad Austral de Chile, Valdivia, Chile.*

e-mails: *canfora@cecs.cl, alexgiacomini@uach.cl*

May 10, 2010

Abstract

In the present paper, a new class of black hole solutions is constructed in even dimensional Lovelock Born-Infeld theory. These solutions are interesting since, in some respects, they are closer to black hole solutions of an odd dimensional Lovelock Chern-Simons theory than to the more usual black hole solutions in even dimensions. This hybrid behavior arises when non-Einstein base manifolds are considered. The entropies of these solutions have been analyzed using Wald formalism. These metrics exhibit a quite non-trivial behavior. Their entropies can change sign and can even be identically zero depending on the geometry of the corresponding base manifolds. Therefore, the request of thermodynamical stability constrains the geometry of the non-Einstein base manifolds. It will be shown that some of these solutions can support non-vanishing torsion. Eventually, the possibility to define a sort of topological charge associated with torsion will be discussed.

1   Introduction

The idea that space-time may have more than four dimensions goes back to Kaluza and Klein in their attempt to find a geometrical unification of gravity with electromagnetism [1]. The idea of extra dimensions is also very natural

in the context of string theory (as a classical textbook on the subject see e.g.[2]) and braneworld scenarios [3]. Thus, the question arises of which is the most general classical theory of gravity in dimension higher than four. To answer this question, one can use a gravitational action which is constructed according to the same criteria as the action of gravity general relativity in four dimensions: one requires that the action must be built from curvature invariants which lead to second order differential equations for the metric. Unlike the four-dimensional case (in which the only action satisfying the previous conditions is the Einstein-Hilbert action plus a cosmological term), in higher than four dimensions there are many more possibilities: this class of theories is called Lovelock gravity [4]. As it has been recently pointed out, Lovelock theories can provide one with a quite intriguing phenomenological scenario for compactification down to four dimensions [5] which is also able to account for the smallness of the cosmological constant. For these reasons, Lovelock gravities are worth to be further investigated.

A peculiar feature of Lovelock gravities is the richness of black hole solutions they admit. Due to the non-trivial structure of the Lagrangian, Lovelock black holes exhibit different types of behavior when compared to the usual black hole thermodynamics of general relativity (see, for instance, [6]). The focus of the present paper will be both on the construction of some new black hole solutions in Lovelock theory and, mainly, on the analysis of the interesting thermodynamical properties of such solutions as well as on the possibility to provide one with "positive entropy bounds" on the geometry of the base manifold.

An interesting class of Lovelock gravities corresponds to the case in which the coupling constants of the theory are chosen in such a way that all the possible $[(D-1)/2]$ maximally symmetric vacua degenerate. In even dimensions this subclass of Lovelock theories is called Born-Infeld gravity. In odd dimensions this request leads to an enhanced local symmetry group and the theory becomes equivalent to a gauge theory with Chern-Simons action. These two types of theories have very different physical properties. For instance, the black solutions with positive constant curvature base manifolds (Born-Infeld and Bañados-Teitelboim-Zanelli respectively) have different lapse functions and thermodynamical behavior [7] , [8]. Moreover due to the enhanced symmetry of the Chern-Simons theory, the equations of motion for the Black hole metric in odd dimensions leave the base manifold completely arbitrary, while in even dimensions this degeneracy does not occur. On the other hand, it is also known that in Lovelock gravity the base manifold needs to be nei-

ther constant curvature nor Einstein ([9], [10] [11], [12], [13]). However, in the cases of Chern-Simons and Born-Infeld Lovelock theories black hole solutions with non-Einstein base manifold have not received a lot of attention.

The main goal of this paper is to explore the physical properties of black hole solutions in Born-Infeld gravity by allowing the base manifold to be non-Einstein. The most simple case of a non-Einstein manifold is a product of several constant curvature base manifolds. Interestingly enough, in this case the lapse function has not the usual even-dimensional Born-Infeld form as in [7], [8] but the one of a Bañados-Teitelboim-Zanelli (**BTZ**) black hole in odd dimensions. These solutions are therefore in some sense a hybrid between an odd and even dimensional black hole as they have many features in common with odd-dimensional solutions but lack of the high degeneracy of the Chern-Simons case. If one chooses this BTZ-like lapse function, the equations of motion of the theory give a constraint on the geometry of the base manifold which forces it to be non-Einstein. If the base manifold is a product of constant curvature spaces, the equation of motion gives a relation between the curvature radii of the factors. In the case of a constant curvature base manifold, it is always possible to "rescale" the radial coordinate is such a way that the curvature radius can be normalized to ± 1 or zero. In the case of a base manifold which is the product of constant curvature spaces, it is possible to rescale to ± 1 only one of the curvature radii of the constant curvature factors. This introduces a new length scale which affects the thermodynamical behavior of the black holes. In some cases, the entropy becomes a non-positive definite function of the remaining curvature radius. The fact that in Lovelock gravities black holes can have negative entropy was firstly discussed in [14]. In this reference, the authors showed that the arising of negative entropies can be attributed to the freedom in choosing the coupling constants in the Lovelock gravities. However, in the present case, this happens when the Lovelock coupling constants are fixed up to an overall factor in the action in such a way that all the possible $[(D-1)/2]$ maximally symmetric vacua degenerate (note that this choice of the coupling constants does not belong to the cases discussed in [14]). The feature which allows the appearance of negative entropies in the present case is the "non-Einstein nature" of the base manifold. Since, as it has been already noticed in this context [14] and as it is well known in statistical mechanics (the example of mean field spin glass is very clear [15] in this respect), negative entropy is a strong signal of instability, the present results provide one with a thermodynamical bound on the possible geometries of non-Einstein base manifolds.

In this paper, six, eight and ten dimensional examples will be considered in details as they catch all the fundamental features of the higher dimensional cases.

In six dimensions, within the class of base manifolds which are product of constant curvature spaces, the base manifold must have the form $M_2 \times N_2$ where M_2 and N_2 are two-dimensional non-vanishing constant curvature manifolds. In higher than six dimensions, there are more ways to construct base manifolds as product of constant curvature spaces: these new possibilities give rise to different thermodynamical behaviors. In ten dimensions, we will focus on the class of base manifolds with the form $M_4 \times N_4$ (M_4 and N_4 being four-dimensional non-vanishing constant curvature manifolds): in this case, the solution presents different branches which can be compared using thermodynamical arguments. In eight dimensions, a case of special interest is the one in which the six-dimensional base manifold is the product of two non-vanishing constant curvature three dimensional manifolds, M_3 and N_3: the entropy of the corresponding black hole solution identically vanishes. Moreover, with this base manifold, the vacuum solution allows the presence of non-trivial torsion. This is a quite non-trivial result: unless the ansatz for the torsion is chosen carefully as in [16], one gets an over-constrained system of equations which "kills" the torsional degrees of freedom. In odd dimensions, in the case of Chern-Simons (**CS**) Lovelock gravities, the further consistency conditions on the torsion disappear because of the enhanced gauge symmetry of the theory. A black hole with torsion in this case has been constructed in [17]. In even dimensions, the Born-Infeld Lovelock gravity is somehow similar to the CS case since the Lovelock coupling constants are tuned in such a way that there exists only one maximally symmetric vacuum. On the other hand, in the Born-Infeld case, no enhanced gauge symmetry appear and the consistency conditions on the torsion *are not* identically satisfied (anyway, as it has been shown in [18] the Born-Infeld Lovelock gravity is the one where the torsion acquires the maximal number of degrees of freedom). When the torsion lives on a three-dimensional sub-manifold, it can be proportional to the three-dimensional completely antisymmetric tensor whose properties imply that most of the consistency conditions are automatically satisfied. This ansatz, proposed for the first time in [16], allowed to find the first exact vacuum solutions with non-vanishing torsion in a five-dimensional Lovelock

4

theory without enhanced gauge symmetry[1] [19]. For a five-dimensional black hole metric, such ansatz for torsion turns out to be consistent in the CS case [20]. So it is natural to ask if there exist black hole solutions with torsion in the cases in which no enhanced symmetry is present. An interesting possibility explored in [21] is to consider a eight-dimensional metric which represents a five-dimensional black hole with three compact extra dimensions: in this case the metric supports a non-vanishing torsion flux. Instead here we will construct a genuine eight-dimensional black hole with non-zero torsion along the $M_3 \times N_3$ base manifold. Eventually, the present construction strongly suggests that it should be also possible to associate to this torsion a conserved charge closer to a monopole charge than to a Noether charge.

The structure of the paper will be the following: In the second section, the black hole ansatz will be discussed. In the third and fourth sections, the six and eight dimensional cases will be discussed and the corresponding thermodynamical features will be analyzed using the Wald formula [22] [23]. It will be also discussed that, in the eight dimensional case, torsion can be introduced in a very natural way. In the fifth section, the possibility to define a topological charge associated to the torsion is analyzed. In the sixth section the ten dimensional case is considered. In the last section, some conclusions are drawn.

2   The Black Hole ansatz

A newsworthy class of Lovelock theories in D dimensions is the one for which all the possible $[(D-1)/2]$ maximally symmetric vacua degenerate to one vacuum. This corresponds in odd dimensions to a Chern-Simons (CS) theory and in even dimensions to a Born-Infeld (BI) theory. The static black hole solutions with a spherical base manifold are given by

$$ds^2 = -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2 d\Omega^2 \tag{1}$$

Where the function f is given by

$$f^2(r) = r^2/l^2 - \mu \tag{2}$$

---

[1]An amusing feature of this solution is that it can be interpreted as the purely gravitational analogue of the Bertotti-Robinson space-time.

in the odd dimensional case. It is important to stress that in the CS case the equations of motion leave the base manifold arbitrary[2]. In the even dimensional case f is given by

$$f^2(r) = r^2/l^2 + 1 - \frac{2M^{D/2-1}}{r} \quad , \tag{3}$$

where, unlike the case of CS Lovelock gravities, the base manifold is determined by the equations of motion to be (in the case of the lapse function in Eq. (3)) of constant curvature. Thus, in the case of BI Lovelock gravities, static spherically symmetric black holes whose base manifolds have positive constant curvature cannot have the lapse function in Eq. (2). On the other hand, many of the nice features of the thermodynamics of CS Lovelock black holes (see for a nice review [6]) are closely connected to the lapse function in Eq. (2). Therefore, it is very interesting to try to "push" the BI Lovelock theory in even dimensions to see if it is possible to construct some black hole solutions close enough to the BTZ-like black hole solutions of CS Lovelock gravities.

Indeed, this goal can be achieved provided one allows the base manifold to be non-Einstein. The most simple example of a non-Einstein manifold is a product of several constant curvature spaces. Interestingly enough, it will be shown here that in this case the lapse function can have the BTZ-like form in Eq. (2).

The gravitational action of the even dimensional Born-Infeld Lovelock theory in $D = 2n$ dimensions can be written in the first order formalism as

$$L_{2n} = \kappa \int \epsilon_{a_1 \ldots a_{2n}} \overline{R}^{a_1 a_2} \ldots \overline{R}^{a_{2n-1} \, 2n} \tag{4}$$

where we have defined

$$\overline{R}^{a_i a_j} = R^{a_i a_j} + \frac{e^{a_i} e^{a_j}}{l^2} \quad ,$$

$R^{a_i a_j}$ is the curvature two form

$$R^{a_i a_j} = d\omega^{a_i a_j} + \omega^{a_i}{}_{a_k} \omega^{a_k a_j} \quad ,$$

---

[2]However, in five dimensions, at least in the case of zero-mass black hole, the degeneracy can be removed by requiring that the solution preserves half of the supersymmetries of the corresponding Chern-Simons supergravity [20]: this can be only achieved by introducing torsion in a suitable way.

$\omega^{a_i a_j}$ is the spin connection and e^{a_i} is the vielbein and the overall coupling constant κ will be discussed in the next sections. In literature it is customary to call the coupling constant of the n-th order Lovelock term c_n. The Born-Infeld tuning between the coupling constants c_n then, according to (4), is found by just applying Newton's binomial formula. The equations of motion obtained by varying the action with respect to the vielbein and the spin connection read respectively

$$E_{a_i} = \epsilon_{a_1...a_{2n}} \overline{R}^{a_2 a_3} \ldots \overline{R}^{a_{2n-2} \, a_{2n-1}} e^{a_{2n}} = 0 \ . \tag{5}$$

$$E_{a_i a_j} = \epsilon_{a_1...a_{2n}} T^{a_1} \overline{R}^{a_2 a_3} \ldots \overline{R}^{a_{2n-4} \, a_{2n-3}} e^{a_{2n-2}} = 0 \ . \tag{6}$$

In this section it will be considered the torsion free case so that the following identity holds:

$$T^{a_i} = De^{a_i} = de^{a_i} + \omega^{a_i}{}_{a_k} e^{a_k} = 0$$

where T^{a_i} is the torsion field. The ansatz for the black hole metric is

$$ds^2 = -f^2 dt^2 + \frac{dr^2}{f^2} + d\Sigma^2 \tag{7}$$

where Σ is a $2n-2$ manifold which is the product of constant curvature manifolds. The lapse function is of the form

$$f^2 = r^2/l^2 - \mu \ , \tag{8}$$

where μ is a mass parameter since a non-trivial μ allows the presence of a black hole horizon. The vielbein are of the form

$$e^0 = f dt \ ; \quad e^1 = \frac{dr}{f} \ ,$$

$$e^i = r \tilde{e}^i \ ,$$

where $\tilde{e}^i$ are the intrinsic vielbein of the base manifold.

With the ansatz in Eqs. (7) and (8) the nontrivial equations of motion are

$$E_0 = \epsilon_{01 i_1 ... i_{2n-2}} \overline{R}^{i_1 i_2} \ldots \overline{R}^{i_{2n-3} i_{2n-2}} e^1 = 0 \ , \tag{9}$$

$$E_1 = \epsilon_{10 i_1 ... i_{2n-2}} \overline{R}^{i_1 i_2} \ldots \overline{R}^{i_{2n-3} i_{2n-2}} e^0 = 0 \ . \tag{10}$$

In the case of a constant curvature base manifold, the above equations of motion would have forced the mass parameter to be $\mu = -1$ preventing the appearance of a horizon. On the other hand, as it will be shown in the next section, when the base manifold is the product of two (or more) constant curvature manifolds, a black hole horizon can appear.

3 Six dimensional black holes

The simplest example in six dimensions is the case in which the base manifold has the form $M_{(2)} \times N_{(2)}$ (where $M_{(2)}$ and $N_{(2)}$ are manifolds of constant curvatures with curvatures γ and η respectively). The metric reads then

$$ds^2 = -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2(dM_2^2 + dN_2^2) \quad , \tag{11}$$

where dM_2^2 and dN_2^2 are the intrinsic metrics of M_2 and N_2. The indices along the manifold M_2 will be denoted as i, j, k,... while the indices along N_2 will be denoted as a, b, c... With this notation, the only nonzero curvatures are

$$\overline{R}^{ij} = \frac{\gamma + \mu}{r^2}e^i e^j \ , \quad \overline{R}^{ab} = \frac{\eta + \mu}{r^2}e^a e^b \ , \quad \overline{R}^{ia} = \frac{\mu}{r^2}e^i e^a \ ,$$

so that the equations of motion Eqs. (9) and (10) (in the present case the two equations of motion are not independent, we will also assume that $\mu \neq 0$) read

$$(\mu + \gamma)(\mu + \eta) + 2\mu^2 \ = \ 0 \ \Leftrightarrow \tag{12}$$

$$(1 + \frac{\gamma}{\mu})(1 + \frac{\eta}{\mu}) + 2 \ = \ 0 \ , \tag{13}$$

thus, the two relevant parameters are

$$\gamma' = \frac{\gamma}{\mu} \quad and \quad \eta' = \frac{\eta}{\mu}.$$

In the present case it is convenient to rescale the mass parameter μ in the lapse function to 1 since this allows to keep track easily of the curvature scales. Obviously, by fixing the mass parameter μ, *one is not fixing the mass* as the physical mass is also proportional to the volume of the base manifold which depends on the curvature scales γ and η. Note also that the case $\mu = 0$ does not represent a black hole. With this normalization of the mass parameter, the equation of motion reads

$$(1 + \gamma)(1 + \eta) + 2 = 0 \ \Rightarrow \tag{14}$$

$$\gamma = -\frac{(3 + \eta)}{1 + \eta} \ , \quad \eta \neq -1 \ , \quad \gamma \neq -1 \ . \tag{15}$$

In this six dimensional case (in which BI gravity is a particular case of Einstein-Gauss-Bonnet gravity), the expression in Eq. (15) is the first concrete solution of the constraint proposed in [9], [10] [11] for Einstein-Gauss-Bonnet gravity in arbitrary even dimensions in the case of degenerate vacua. Using the above expression for γ, one can easily check that there is no choice of η for which $\gamma = \eta$ in such a way that a BTZ lapse function does not allow Einstein base manifolds. As the radial coordinate has been already used to normalize the mass parameter, the curvature radius η is indeed an integration constant. Looking at the equation (15) it is easy to realize that the possible base manifolds are $H_2 \times H_2$ and $H_2 \times S_2$ while the case of $S_2 \times S_2$ is not realized since γ and η cannot be both positive (here, S_2 represents a two-dimensional compact positive constant curvature space while H_2 represents a two-dimensional compact negative constant curvature space).

It is worth pointing out that for base manifolds which are the product of constant curvature spaces *in which there is only one factor with non-vanishing curvature* one would have the curvature radius and the mass completely fixed by the equations of motion. Such isolated points in the solution space are of less physical interest as in such a case it is not obvious how to achieve a satisfactory thermodynamical description. Eventually, a base manifolds which is a product of vanishing curvature spaces is not compatible with the ansatz in Eqs. (7) and (8).

As far as the class of metrics which we are analyzing is concerned, the solutions must have a base manifold where at least two factors have non-trivial curvature. This means that in six dimensions the base manifolds $H_2 \times H_2$ or $H_2 \times S_2$ are the only possibilities. The intrinsic volume of the corresponding base manifold (which is important in order to compute the mass) reads

$$Vol\left(M_2 \times N_2\right) = \xi \frac{(1+\eta)^2}{\eta^2(3+\eta)^2} \tag{16}$$

where ξ is a positive combinatorial factor which will not play any role in the following. The entropy of this solution can be computed using the Wald formula [22]. For Lovelock theories, there is a simple way to apply the Wald formula [23]: the n-th Lovelock term "$c_n R^n$" in D dimensions (which has n factors of the curvature two form) in the action ,

$$c_n R^n \approx c_n \epsilon_{a_1..a_D} R^{a_1 a_2}..R^{a_{2n-1} a_{2n}}..e^{a_D} \ ,$$

(c_n being the coupling constant of the $n-$th term in the Lovelock series)

contributes to the entropy with a term $S^{(n)}$ of the form

$$S^{(n)} = nc_n \int_H \epsilon_{a_1..a_{D-2}} R^{a_1 a_2}..R^{a_{2n-3} a_{2n-2}}..e^{a_{D-2}} \tag{17}$$

where the integral is evaluated on the horizon H. Thus, using the BI tuning for the coupling constants, the contribution $S^{(n)}$ depends on the $(n-1)$-th Lovelock term evaluated on the horizon H. Then, the total Wald entropy reads

$$S = \sum S^{(n)} . \tag{18}$$

In six dimensions the highest nontrivial Lovelock term is the quadratic Gauss-Bonnet term while, as usual, the contribution of the Einstein-Hilbert term is proportional to the volume of the base manifold: thus, the total entropy in Eq. (18) reads

$$S_{6D} = S_{6D}(\eta) = \kappa c_1 r_+^4 Vol\left(M_2 \times N_2\right) \left[\frac{2(\eta^2 - 3)}{1 + \eta} + 6 \right] , \tag{19}$$

where r_+ is the radius of the black hole horizon (which in our units is $r_+ = |l|$). Inserting the explicit expression for the intrinsic volume of the base manifold and (13) the entropy becomes

$$S_{6D} = 2\kappa\xi r_+^4 c_1 \frac{\eta + 1}{(\eta + 3)\eta} \tag{20}$$

It is worth pointing out that the above expression for the entropy in this six-dimensional case, presents a quite interesting and novel feature. The entropy has a zero in $\eta = -1$ where it changes sign. When the overall coupling constant κ of the six-dimensional BI-Lovelock action is chosen to be positive (as it is usual [7]), the entropy in Eq. (20) is negative for $\eta < -3$ and $-1 < \eta < 0$. These ranges for the integration constant correspond to a base manifold of the form $H_2 \times H_2$. In the present case, choosing a negative κ would not help to avoid the arising of a negative entropy[3]. The appearance

---

[3]In classical thermodynamics the entropy is defined up to an additive constant. Therefore, one may wonder whether a suitable additive constant can be found in order to avoid the appearence of negative entropy solutions. On the other hand, as it can be seen in Eq. (20), in the present cases the entropy can even approach to $-\infty$. Thus, it seems that the appearence of negative entropy solutions cannot be avoided by adding a suitable constant to the (Wald) entropy.

of black hole solutions with negative entropy in higher order theories has been already pointed out in [14]: as known results in statistical mechanics suggest (see, for instance, the classic review on spin glass [15]), a solution with negative entropy should decay into a more stable solution. In [14], the origin of the negativity of the entropy was the possibility to consider cases in which some of the coupling constants of the theory can be arbitrary negative. In the present example, the coupling constants are the ones characterizing the BI Lovelock Lagrangian and, in particular, this choice of the coupling constants appears to be outside the range of [14]. Therefore, in the present example the non-Einstein nature of the base manifold (and in particular the appearance of a further curvature scale) is behind the sign change of the entropy. Hence, the above thermodynamical argument provides the geometry of the base manifold with a stability constraint. A necessary condition in order for the black hole in Eqs. (11), (8) and (15) to be thermodynamically stable is that when $\kappa > 0$ the curvature η of the factor $N_{(2)}$ of the base manifold has to satisfy the following *positive entropy bound*

$$\eta > 0 \quad or \quad 3 < \eta < -1$$

This implies that the base manifold must be of the form $S_2 \times H_2$ (the base manifold $S_2 \times S_2$ was already excluded by the equations of motion). If $\kappa < 0$, one has to consider the complement of the above conditions. Eventually, the mass of the black hole can be computed using the first law of black hole thermodynamics which in the static case takes the usual form [23] $\delta U = T \delta S$ where the temperature T is given as usual in terms of the derivative of the lapse function at the horizon $T = \frac{1}{4\pi} f'$. As it is natural to expect, it is indeed proportional to the intrinsic volume of the base manifold. In any case, in the following analysis, it will be enough to study the behavior of the Wald entropy.

4   Eight dimensional black hole

In eight dimensions more possibilities for the choice for the base manifolds appear. Also in this case, it is convenient to normalize to one the mass parameter $\mu = 1$ since the mass parameter will be assumed to be non-vanishing. The most interesting case is $M_3 \times N_3$ where the two factors are three dimensional constant curvature manifold with (non-vanishing) curvatures γ and η.

The reason is that, with this choice, the solution may also support a non-trivial torsion flux and exhibits a very peculiar thermodynamics as it will be explained in the next section. The metric reads

$$ds^2 = -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2(dM_3^2 + dN_3^2) \ , \tag{21}$$

where the lapse function is as in Eq. (8). The equations of motion with the useful normalization $\mu = 1$ gives

$$3(\gamma + 1)(\eta + 1) + 2 = 0 \quad \Rightarrow \quad \gamma = -\left(\frac{3\eta + 5}{3(\eta + 1)}\right) \ . \tag{22}$$

Using the above expression for γ, one can easily check that also in this case there is no choice of η which allows $\gamma = \eta$. In other words, if the lapse function has the BTZ form then the base manifold cannot be Einstein. The intrinsic volume of the base manifold is

$$Vol(M_3 \times N_3) = \xi \frac{1}{|\gamma\eta|^3} = \xi \left|\frac{27(\eta + 1)^3}{\eta^3(3\eta + 5)^3}\right|$$

where ξ is a positive combinatorial factor which will not be important for the following analysis. Recalling that in eight dimensions the highest non-trivial Lovelock term is cubic in the curvature, Eqs. (17), (18), leads to the following expression for the Wald entropy

$$\begin{aligned} S &= Vol(M_3 \times N_3)(r_+)^6 \left[3\frac{c_3}{l^4}(4\gamma\eta) + \right. \\ &\quad \left. +2\frac{c_2}{l^2}(4\gamma + 4\eta) + 20c_1\right] \ . \end{aligned}$$

Using the BI tuning of the Lovelock coupling constants in eight dimensions,

$$\frac{c_3}{c_2} = \frac{2l^2}{3} \ , \quad \frac{c_1}{c_2} = \frac{2}{3l^2} \ ,$$

as well as Eq. (22), it is easy to see that the entropy of this black hole identically vanishes for any value of the integration constant η

$$S \equiv 0 \ .$$

It is worth to point out that the temperature of this black hole remains finite. Cases of black holes with zero entropy and finite temperature are known in

12

literature in the context of compactified Lovelock black holes as well as in the context of Lifshitz black holes [24] [25][4]. In these cases the vanishing of the entropy is related to a specific tuning of the coupling constants rather than to the topology of the base manifold. For the class of black hole solutions studied in the present paper, the existence of a solution with zero entropy is due to the special topology (in a sense that will be explained in the next section) of the base manifold $M_3 \times N_3$.

4.1 $M_4 \times N_2$ base manifold

A black hole solution with zero entropy is a quite peculiar feature of the $M_3 \times N_3$ base manifold. This can be seen as follows. In eight dimensions there is, for instance, the possibility to choose a less symmetric base manifold such as $M_4 \times N_2$. Let the curvature radius of the M_4 factor be γ and the one of N_2 factor be η. The equations of motion imply the relation

$$\eta = -\frac{5 + \gamma}{1 + \gamma} \tag{23}$$

The entropy of the corresponding black hole reads

$$S = 3\kappa\xi r_+^6 c_1 \frac{(1+\gamma)^2}{\gamma^4(5+\gamma)^2} \left[\gamma^2 + 6\gamma - \frac{5+\gamma}{1+\gamma}(2\gamma + 1) + 5 \right] \tag{24}$$

(ξ being a positive combinatorial factor which plays no role in the present analysis) which is negative for $-5 < \gamma < -1$. This range of parameters corresponds to a base manifold of the form $H_4 \times S_2$. It is worth to point out that the entropy corresponding to the base manifold $S_4 \times H_2$ is positive. Thus, in particular, the entropy in this case is not identically zero. Analogous computations for a more general choice of base manifolds as product of constant curvature spaces reveal that an identically vanishing entropy is a peculiarity of the $M_3 \times N_3$ base manifold.

5   non-trivial torsion

Besides the identically vanishing entropy, the eight dimensional black hole with base manifold $M_3 \times N_3$ has another very interesting feature: namely it

---

[4]Other papers somehow related to this subject are [26].

supports a non trivial torsion flux according to the prescription of [16]. In particular, this appears to be the first black hole solution with non-vanishing torsion[5] in the case of a Lovelock theory without enhanced gauge symmetry. The following ansatz for the torsion

$$T^i = F_1(r)\epsilon^{ijk}e_k e_k \ , \quad T^a = F_2(r)\epsilon^{abc}e_b e_c \ , \tag{25}$$

and the contorsion tensor

$$K^{ij} = -F_1(r)\epsilon^{ijk}e_k \quad ; \quad K^{ab} = -F_2(r)\epsilon^{abc}e_c \tag{26}$$

with

$$F_1 = \frac{\delta_1}{r} \quad ; \quad F_2 = \frac{\delta_2}{r} \tag{27}$$

is very natural since the only modifications of the curvature two form (which now receives contributions also from the contorsion tensor) are

$$R^{1i} = \hat{R}^{1i} - \frac{f}{r}T^i \quad ; \quad R^{ij} = \hat{R}^{ij} - \left(\frac{\delta_1}{r}\right)^2 e^i e^j \tag{28}$$

$$R^{1a} = \hat{R}^{1a} - \frac{f}{r}T^a \quad ; \quad R^{ab} = \hat{R}^{ab} - \left(\frac{\delta_2}{r}\right)^2 e^a e^b \quad \Rightarrow \tag{29}$$

$$R^{ij} = \frac{\tilde{\gamma} - f^2}{r^2}e^i e^j \ , \quad R^{ab} = \frac{\tilde{\eta} - f^2}{r^2}e^a e^b \ , \tag{30}$$

$$\tilde{\gamma} = \gamma - (\delta_1)^2 \ , \quad \tilde{\eta} = \eta - (\delta_2)^2 \ , \tag{31}$$

where γ and η denote the Riemannian torsion-free parts of the scalar curvatures of the two factors of the base manifold, $\hat{R}^{ab}$, $\hat{R}^{ij}$, $\hat{R}^{1i}$, $\hat{R}^{1a}$,... denote the Riemannian torsion-free parts of the curvature while R^{ab}, R^{ij}, R^{1i}, R^{1a},... denote the total curvature. It is easy to see that with this choice of the torsion, the equations of motion obtained by varying the spin connection for the torsion (6) are identically satisfied so that δ_1 and δ_2 are integration constants.

---

[5]It is worth noting that, in [21], it has been constructed a compactified black hole solution with torsion in eight dimensions: namely an eight-dimensional metric which represent a five-dimensional black hole with three compactified extra dimensions. While the black hole solution with torsion found in [20] corresponds to a Chern-Simons Lovelock theory with, therefore, enhanced gauge symmetry.

The equations of motion obtained by varying the action with respect to the vielbein (5) reduce once again to Eq. (22) *but with the replacement*

$$\gamma \to \tilde{\gamma} = \gamma - (\delta_1)^2 \quad , \quad \eta \to \tilde{\eta} = \eta - (\delta_2)^2 \ . \tag{32}$$

The technical reason for this lies in the nice identities satisfied by a torsion of the form in Eq. (25): see, for instance, [16] and [21]. Interestingly enough, as it will be explained in more details in the next section, the geodesics of the metric in Eq. (21) with no-vanishing torsion fluxes as in Eqs. (25) and (27) along the two factors $M_{(3)}$ and $N_{(3)}$ "almost" do not feel the torsion. Namely, the torsion enters the metric only in Eq. (22) through the replacements in Eq. (32) which affects the relation between γ and η:

$$\tilde{\gamma} = -\left(\frac{3\tilde{\eta} + 5}{3(\tilde{\eta} + 1)}\right) \quad \Leftrightarrow \quad \gamma = (\delta_1)^2 - \left(\frac{3\left(\eta - (\delta_2)^2\right) + 5}{3(\eta - (\delta_2)^2 + 1)}\right) .$$

On the other hand, being the torsion fully anti-symmetric, the corresponding contorsion tensor does not affect directly the geodesic equation: see, for instance, [21]. However, the dynamics of Fermions is affected by the presence of torsion: the contorsion tensor directly enters the Dirac equation modifying the spin connection in a way very similar to a "spin-spin" coupling in which the contorsion plays the role of "spin" of the gravitational background. At present, a formulation of Wald entropy in the presence of non-vanishing torsion fluxes is still unavailable but it can be argued that torsion could also have, for suitable choices of the integration constants δ_1 and δ_2, positive effects on the stability of the solution (as it was first shown in [20]). A more detailed discussion on this point is in the following section.

5.1   topological charge?

It is very interesting to discuss how the presence of torsion may affect the definition of charges and, in particular, the possibility to construct some sort of topological charge: this issue is closely connected to the question of stability. In general, one should expect that the torsion affects the definition of conserved charges in (at least) two ways.

Firstly, when defining the conserved charges in Lovelock gravities using asymptotic integrals and counterterms [27] [28] or using the modified version of the Komar integrals [29] one always uses the absence of torsion to derive

15

some useful identities (in a recent paper [30] the author has shown how to include possible torsion contributions). For this reason, in the presence of a non-trivial torsion, one should expect that the usual charges get some extra contributions.

Secondly and more interestingly from the point of view of the stability analysis, one should also expect that the presence of torsion could give rise to some sort of *topological charge*.

To clarify this point, let us briefly recall some well known features of Yang-Mills-Higgs theory. In this case, as it is well known, the (non-Abelian generalization of the) electric charges can be constructed, for instance, using the Noether method. In order to perform such procedure, one needs to use both the symmetries of the action and the equations of motion. On the other hand, in non-Abelian Yang-Mills-Higgs theory it is also possible to define a monopole charge. The monopole charge is indeed conserved (nice detailed reviews are, for instance, [31] [32] [33]) but, in order to prove this fact, one needs to use the *boundary conditions and the Bianchi identities* (instead of the equations of motion). A similar situation occurs in scalar field theories in two dimensions which admit kinks solutions: the kink number can be defined and one can prove that the kink number is conserved using just the boundary conditions but without using the equations of motion and the symmetries of the corresponding action. For these reasons, the term "topological charge" is commonly used in these cases. Furthermore, in the cases in which the gauge theory admits a supersymmetric version, topological charges play an important role since they enter the so called BPS bound (see, for instance [33]) which represents a (necessary) condition in order for classical solutions to preserve a fraction of the supersymmetries of the theory. Roughly speaking, the BPS bound can be written as

$$M \geq |Q_t| \tag{33}$$

where M is the mass of the classical solution and Q_t is the corresponding topological charge. When the bound is saturated one can expect that the classical solution preserves some fraction of the supersymmetries. As it is well known, BPS-like bounds play a very important role in the stability analysis of classical backgrounds: even in situations in which no supersymmetric argument is available, a BPS-like bound can help in proving the stability of the corresponding solution.

In the case of Lovelock gravities, there is a close analogy between the role of torsion and the non-Abelian magnetic field of a Yang-Mills theory which

suggests a natural ansatz (first proposed in [16]) to find exact solutions with non-trivial torsion. Thus, inspired by the case of Yang-Mills-Higgs theories in which the BPS bound often implies some linear relations among the electric and magnetic part of the field strength, the idea is to consider a torsion which is a linear combination of the components of the curvature which appears in the equations of motion (see Eq. (25)). This idea allowed to construct the first exact vacuum solutions with non-trivial torsion in Lovelock gravities [19] [20] [21]. In particular, it has been shown in [19] that using the proposed ansatz for the torsion it is possible to construct a class of black hole solutions in five dimensional Chern-Simons supergravity with a ground state which preserves half of the supersymmetries (something which, without torsion is known to be impossible). Therefore, being the torsion necessary in order to achieve a half-BPS black-hole solution, it is quite natural to expect that it should be possible to construct with torsion a topological charge (namely, a charge which is constructed using the Bianchi identities but without using the equations of motion or the symmetries of the solution) which enters some sort of BPS bound as in Eq. (33).

In order to provide one with a more concrete argument supporting the fact that torsion could lead to the definition of a topological charge, one can proceed as follows. We will construct for the special class of metric in Eq. (21) a six-form Q_Ω such that DQ_Ω vanishes by virtue of only the Bianchi identities.

It is useful to recall here the Bianchi identities corresponding to a generic $N-$dimensional background metric with torsion (whose curvature two-form and torsion two form read R^{AB} and T^C with $A, B, C = 1, ..., N$):

$$DT^C = R_{CB}e^B , \quad DR^{AB} = 0 , \tag{34}$$

where e^B is the vielbein and D is the covariant derivative.

Let us consider the two following three-forms

$$\Omega_{(1)} = H(r)T^a e_a , \quad \Omega_{(2)} = H(r)T^i e_i , \tag{35}$$

where $\tilde{e}_i$ and $\tilde{e}_a$ are the intrinsic vielbeins of M_3 and N_3 and

$$H(r) = \frac{1}{r^2} .$$

Using both the Bianchi identities in the presence of torsion and the explicit expressions of the curvature in the presence of non-vanishing torsion in Eqs.

17

(28) and (29) (*but without using the equations of motion*), one easily gets that the six-form Q_Ω, defined as

$$Q_\Omega = \Omega_{(1)} \wedge \Omega_{(2)} \quad , \tag{36}$$

is closed

$$DQ_\Omega = 0 \ .$$

Indeed, this property of the six-form Q_Ω is similar to the property which is needed to construct a conserved charge in Lovelock theories in the Komar approach [29]. The usual strategy is to evaluate the integral of a closed form on a spacelike slice of the space-time in such a way to get two contributions, one coming from the asymptotic region and another coming from the horizon. Then, using the fact that the form is closed, one could conclude that the two contributions are equal obtaining as a by-product the conserved charge as well. Interestingly enough, using the expressions in Eq. (36) one would obtain a charge which is proportional to the product $\delta_{(1)}\delta_{(2)}$ where the integration constants $\delta_{(1)}$ and $\delta_{(2)}$ correspond to the presence of non-trivial torsion fluxes along both N_3 and M_3. It is worth noting that the present proposal in Eq. (36) would give a vanishing topological charge if only one of the constants ($\delta_{(1)}$ or $\delta_{(2)}$) would be different from zero. In order to go through with this method for the definition of the topological charge associated with torsion, one should construct suitable forms associated with torsion similar to the ones in Eq. (36) which are closed due to *only* the Bianchi identities for a generic vacuum solution with torsion of Lovelock theories and not just for the class of metrics in Eq. (21). Because of the scarcity of exact solutions with torsion in Lovelock theories, this task seems to be quite difficult but it certainly deserves further investigation.

A natural question (which is also closely related to the nature of the would be topological charge which has been discussed above) is: what kind of physical effects are expected in a background in which the torsion has the form in Eq. (25)? Indeed, it can be easily seen that geodesics, in a background metric in Eq. (21) with a totally anti-symmetric torsion as described in Eq. (25), are not affected by the presence of torsion. On the other hand, the dynamics of spinors is affected by the presence of torsion since the contorsion enters directly into the Dirac equations. Thus, as it has been pointed out in [21], in a background with torsion the Dirac fields are, in a sense, "polarized" by the torsion itself. For this reason, if it would be possible to define in general a topological charge associated with (the totally antisymmetric part of)

torsion in Lovelock gravities then the nature of this charge would be more like a spin than as a usual angular momentum.

6   D=10 base manifold $M_4 \times N_4$

In order to emphasize the peculiarity of the base manifold $M_3 \times N_3$ in the eight dimensional black hole solution, it is worth to analyze here its most natural generalization to ten dimensions. In particular we will show that in the ten dimensional case the entropy is not identically zero. In this case the base manifold is the product of two four dimensional constant curvature manifolds with curvature γ and η. Once again, it is convenient to normalize the mass parameter μ to one: $\mu = 1$. Then, the equations of motion give

$$3(\gamma + 1)^2(\eta + 1)^2 + 24(1 + \gamma)(1 + \eta) + 8 = 0 \tag{37}$$

An interesting feature of this case is that Eq. (37) is quadratic in η and γ. Therefore, for a fixed η, there are two possible values $\gamma_\pm$ of γ:

$$\gamma_\pm = -1 - \frac{12(1 + \eta) \mp |1 + \eta| \sqrt{120}}{3(\eta + 1)^2}$$

so the solution admits two branches. The fact that solutions of Lovelock gravity generally admit several branches is well known [34]and due to the fact that the theory is polynomial in the curvature. In this context naturally arises the question if the two branches behave differently from the thermodynamical point of view.

The Wald entropy can be easily computed as function of η and γ and reads

$$S = \kappa r_+^8 c_1 Vol(\Sigma) \left[12(\gamma^2\eta + \eta^2\gamma) + 6(\gamma^2 + \eta^2 + 12\gamma\eta) + 60(\gamma + \eta) + 70 \right] \tag{38}$$

In order to evaluate the entropy for the two branches it is necessary to insert the corresponding expressions in Eq. (39). It can be seen that for both branches the behavior is qualitatively identical and especially the entropy does not change sign in both branches. Depending on the sign of the overall coupling constant κ, the entropy can be negative definite or positive definite. In the four-dimensional case, the overall factor κ has to be chosen to be positive since, when $D = 4$, a negative κ would imply that the graviton is a ghost. On the other hand, in dimensions higher than four, the BI Lovelock theory is

degenerate when expended around the maximally symmetric vacuum so that it is difficult to define a meaningful graviton propagator so, when $D > 4$, one cannot use the same argument as in $D = 4$ to fix the sign of κ. Indeed, as it has been first noticed in [5], it is even possible to take advantage of the freedom to choose the overall sign in Lovelock gravities to provide one with natural mechanism for spontaneous compactification. Therefore, in the present case, it is possible to choose the constant κ to be negative in order obtain a positive definite entropy. The possibility of a negative overall factor for a gravitational action, in order to get a positive entropy, was argued also in [25] in the context of the 3-D Lifshitz black hole [35].

7 Discussion and conclusions

In the present paper, a new class of black hole solutions has been constructed in even dimensional Lovelock Born-Infeld theory. These solutions are interesting since, in some respects, they are closer to black hole solutions of an odd dimensional Lovelock Chern-Simons theory than to the more usual black hole solutions in even dimensions: this hybrid behavior arises when non-Einstein base manifolds are considered. The thermodynamical features of these solutions have been analyzed using Wald entropy. It turns out that depending on the geometry of the base manifold the corresponding entropy can change sign and even be zero with, at the same time, finite temperature. Similar examples were already known in the literature on higher order gravity theories. However, in such examples the origin of this behavior lies in the choice of the coupling constants. In the present case this peculiar behavior is due to the non-Einstein nature of the base manifold. Indeed, when the base manifold is the product of different constant curvature spaces, a new curvature scale appears which plays the role of an integration constant. The sign of the entropy then depends on the value of the integration constant. In this way the request of thermodynamical stability constrains the geometry of the base manifold. It has been also shown that in eight dimensions when the base manifold has the form $M_3 \times N_3$ (which leads to a vanishing entropy) a non-vanishing torsion flux can appear. Eventually, the possible role of torsion in the definition of a topological charge in Lovelock gravity has been briefly discussed.

Acknowledgements

References

[1] T. Kaluza: Zum Unitaritätsproblem der Physik. Sitz. Preuss. Akad Wiss. Phys. Math. K1 (1921) 966;
O. Klein: Quantentheorie und Fünfdimensionale Relativitätstheorie. Zeits. Phys. 37 (1926) 895

[2] M.B. Green, J.H. Schwartz, E. Witten: Superstring theory (Volume I and II). Cambridge University Press (1987)

[3] L. Randall and R. Sundrum: A Large mass hierarchy from a small extra dimension. Phys.Rev.Lett.83 (1999)

[4] D. Lovelock, J. Math. Phys. **12**, 498 (1971).

[5] F. Canfora, A. Giacomini, R. Troncoso, S. Willison, *Phys.Rev.* **D80** (2009), 044029.

[6] C. Garraffo, G. Giribet, *Mod. Phys. Lett.* **A23** (2008), 1801.

[7] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. D **49**, 975 (1994)

[8] J. Crisostomo, R. Troncoso and J. Zanelli, Phys. Rev. D **62**, 084013 (2000)

[9] G. Dotti, J. Oliva and R. Troncoso, Phys. Rev. D **76**, 064038 (2007)

[10] G. Dotti, J. Oliva and R. Troncoso, Int. J. Mod. Phys. A **24**, 1690 (2009)

[11] G. Dotti, J. Oliva, R. Troncoso, *"Static solutions with nontrivial boundaries for the Einstein-Gauss-Bonnet theory in vacuum"* arXiv:1004.5287.

[12] C. Bogdanos, C. Charmousis, B. Gouteraux, R. Zegers, *JHEP* 0910: 037 (2009).

[13] H. Maeda, *"Gauss-Bonnet black holes with non-constant curvature horizons"* arXiv:1004.0917.

[14] M. Cvetic, S. Nojiri and S. D. Odintsov, Nucl. Phys. B **628**, 295 (2002)

[15] M. Mezard, G. Parisi, M. Virasoro, *"Spin Glass Theory and Beyond"*, Lecture Notes in Physics (Vol 9), World Scientific (1987).

[16] F. Canfora, *Some solutions with torsion in Chern-Simons gravity and observable effects*, arXiv:0706.3538.

[17] R. Aros, M. Contreras, Phys.Rev. D73 (2006) 087501.

[18] R. Troncoso, J. Zanelli, *Class.Quant.Grav.* 17 (2000) 4451.

[19] F. Canfora, A. Giacomini and S. Willison, Phys. Rev. D **76**, 044021 (2007)

[20] F. Canfora, A. Giacomini, R. Troncoso, *Phys. Rev.* **D77** (2008) 024002.

[21] F. Canfora, A. Giacomini, *Phys. Rev.* **D78** (2008) 084034.

[22] T. Jacobson, G. Kang and R. C. Myers, Phys. Rev. D **49**, 6587 (1994)

[23] T. Jacobson and R. C. Myers, Phys. Rev. Lett. **70**, 3684 (1993)

[24] R. G. Cai, L. M. Cao and N. Ohta, Phys. Rev. D **81**, 024018 (2010)

[25] R. G. Cai, Y. Liu and Y. W. Sun, JHEP **0910**, 080 (2009) R. G. Cai, Y. Liu and Y. W. Sun, JHEP **0910**, 080 (2009).

[26] Rong-Gen Cai, Kwang-Sup Soh, *Phys.Rev.* **D59**: 044013,1999; R. G. Cai, *Phys. Lett.* **B 582**, 237 (2004); R. G. Cai, *Phys. Rev.* **D65** 084014 (2002); R. G. Cai, Q. Guo, *Phys. Rev.* **D69** 104025 (2004).

[27] T. Regge and C. Teitelboim, Annals Phys. **88**, 286 (1974).

[28] R. Aros, M. Contreras, R. Olea, R. Troncoso and J. Zanelli, Phys. Rev. D **62**, 044002 (2000).

[29] D. Kastor, Class. Quant. Grav. **25**, 175007 (2008).

[30] E. Gravanis, *Phys. Rev.* **D 81**, 084013 (2010).

[31] A. Vilenkin, E. P. S. Shellard, *Cosmic Strings and Other Topological Defects*, Cambridge University Press (2000).

[32] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena,* Oxford University press, (2002).

[33] D. I. Olive, P. C. West, *Duality and Supersymmetric Theories,* Cambridge University Press (Publications of the Newton Institute, 2009).

[34] D. G. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985).

[35] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, Phys. Rev. D **80**, 104029 (2009)